**Question 1.** ($a$) Define the notions of sequential compactness and limit point compactness. ($b$) Prove that the Lebesgue covering lemma holds for a sequentially compact metric space.

**Answer.**

($a$) **Sequentially compact:** Let $T = (S, \tau)$ be a topological space and $X \subseteq S$. Then $X$ is called *sequentially compact* in $T$ if every infinite sequence in $X$ has a sub-sequence which converges to a point in $X$.

**Limit point compact:** Let $T = (S, \tau)$ be a topological space and $X \subseteq S$. Then $X$ is called *limit point compact* or weakly countably compact if every infinite subset of $X$ has a limit point in $X$.

($b$) **Lebesgue covering lemma:** If the metric space $(X, d)$ is compact and an open cover of $X$ is given, then there exists a number $\delta > 0$ such that every subset of $X$ having diameter less than $\delta$; is contained in some member of the cover.

Now, we prove that Lebesgue covering lemma holds for a sequentially compact metric space.

**Proof:** Let $\mathcal{U}$ be an open cover of $X$ and suppose, to the contrary, that there is no such $\delta$. Then for every $n \in \mathbb{N}$, $1/n$ is not a Lebesgue number, that is, there is a subset $S_n \subset X$ such that $diam(S_n) < 1/n$ but $S_n$ is not entirely contained in any of the elements of $\mathcal{U}$. (In particular, $S_n$ is not empty.) For each $n \in \mathbb{N}$, we choose one point $x_n \in S_n$. Since $X$ is sequentially compact, there is a subsequence $(x'_n)$ of $(x_n)$ which converges to some point $x \in X$. Since $\mathcal{U}$ covers $X$, there is a $U \in \mathcal{U}$ such that $x \in U$. Then there is an $\epsilon > 0$ such that $x \in N_X(x; \epsilon) \subset U$. Choose $N \in \mathbb{N}$ with $1/N < \epsilon/2$. Since $(x'_n)$ converges to $x$, all but finitely many members of the subsequence $(x'_n)$ must lie in $N_X(x; \epsilon/2)$. Hence, infinitely many members of the original sequence $(x_n)$ lie in $N_X(x; \epsilon/2)$. So, there is an $n > N$ such that $x_n \in N_X(x; \epsilon/2)$. This implies, however, that $S_n \subset N_X(x; \epsilon/2)$. For if $s \in S_n$, then $d(s; x) \leq d(s; x_n) + d(x_n; x) < diam(S_n) + \epsilon/2 < 1/n + \epsilon/2 < 1/N + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$. But then $S_n \subset N_X(x; \epsilon/2) \subset U$, which contradicts our assumption. Therefore, Lebesgue covering lemma holds for a sequentially compact metric space.

**Question 2.** ($a$) Define the notions of a second countable, Lindelöf and separable topological space. ($b$) Prove or disprove: $\mathbb{R}_l$ (the reals with the lower limit topology) is second countable.

**Answer:**

($a$) **Second countable:** Let $T = (S, \tau)$ be a topological space. The topological space $T$ is said to be second-countable if its topology has a countable base. More explicitly, this means that the topological space $T$ is second countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of $T$ such that any open subset of $T$ can be written as a union of elements of some subfamily of $\mathcal{U}$.

**Lindelöf space:** A topological space $T$ is said to be a Lindelöf topological space if every open cover has a countable subcover.

**Separable space:** A topological space $T$ is said to be a separable topological space if $T$ contains a countable dense subset, i.e., there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

($b$) Let $\tau$ be the lower-limit topology on $\mathbb{R}$ and $T = (\mathbb{R}, \tau)$ be the topological space. We will show that $T$ is not second countable. Let $B$ be a base for $\tau$. We shall show that $B$ can not be countable. For each $x$ in $\mathbb{R}$, we choose a basic open set $B_x$ from $B$ such that $x \in B_x \subset (x-1, x]$. This can be done, because $(x-1, x]$ is an open set containing $x$. Now, let $x, y \in \mathbb{R}$ be such that $x < y$. Then $y \notin B_x$ and $y \in B_y$. So $B_x$ is a different set from $B_y$ for all $x < y$. Thus, $B$ is uncountable

**Question 3.** ($a$) Define the notions of a normal, regular, and completely regular topological space. ($b$) State Urysohn's Lemma, Urysohn's Metrization Theorem and the Tietze Extension Theorem. ($c$) Give an example (with details) of a topological space for which Urysohn's lemma does not hold.

(*a*) **Normal topological space:** A topological space $X$ is a normal space if, given any disjoint closed sets $E$ and $F$, there are open neighborhoods $U$ of $E$ and $V$ of $F$ that are also disjoint.

**Regular topological space:** A topological space $X$ is a regular space if, given any closed set $F$ and any point $x$ that does not belong to $F$, there exist a neighborhood $U$ of $x$ and a neighborhood $V$ of $F$ that are disjoint.

**Completely regular topological space:** A topological space $X$ is a completely regular space if, given any closed set $F$ and any point $x$ that does not belong to $F$, there exists a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$.

(*b*) **Urysohn's Lemma:** If $A, B$ are disjoint closed sets in a normal space $X$, then there exists a continuous function $f : X \to [0,1]$ such that $\forall a \in A, f(a) = 0$ and $\forall b \in B, f(b) = 1$.

**Urysohn's Metrization Theorem:** Every regular space $X$ with a countable basis is metrizable.

**Tietze Extension Theorem:** Let $X$ be a normal space and $A$ be a closed subset in $X$. If $f : A \to [a,b]$ is a continuous function, then $f$ has a continuous extension $\bar{f} : X \to [a,b]$, i.e., $\bar{f}$ is continuous and $\bar{f}|_A = f$.

(*c*) Let $X = \{a,b,c\}$ be a space with topology $\tau = \{\emptyset, \{b\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$. It is easy to see that $\tau$ is a topology on $X$ and $\{a\}, \{c\}$ are two closed sets in $X$. Now, we show that there is no continuous function $f : X \to [0,1]$ such that $f(a) = 0$ and $f(c) = 1$. Let $f : X \to [0,1]$ be a function such that $f(a) = 0$ and $f(c) = 1$.

Case 1: Let $f(b) \leq 0$. Then $f^{-1}(0,\infty) = \{c\}$. Since $\{c\} \notin \tau$, $f$ is not continuous.

Case 2: Let $0 < f(b) < 1$. Then $f^{-1}(f(b),\infty) = \{c\}$. Since $\{c\} \notin \tau$, $f$ is not continuous.

Case 3: Let $f(b) \geq 1$. Then $f^{-1}(-\infty,1) = \{a\}$. Since $\{a\} \notin \tau$, $f$ is not continuous.

Thus, Urysohn's Lemma does not hold in this topological space.

**Question 4.** (*a*) Prove or disprove: a locally compact Hausdorff space is completely regular. (*b*) Prove that the one point compactification of the natural numbers $\mathbb{N}$ with the discrete topology is homeomorphic to the subspace $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$.

(*a*) For any topological space $X$ we have the following implications: $X$ is locally compact Hausdorff $\Longrightarrow$ $X$ is an open subspace of a compact Hausdorff space (by Corollary 29.4) $\Longrightarrow$ $X$ is a subspace of a normal space (by Theorem 32.2) $\Longrightarrow$ $X$ is a subspace of a completely regular space (by Theorem 33.1) $\Longrightarrow$ $X$ is completely regular (by Theorem 33.2).

(*b*) The one-point compactification of $\mathbb{N}$ consists of $\mathbb{N}$ together with a single point which we can call $\infty$. The topological structure is that of the discrete topology on $\mathbb{N}$ and the open neighbourhoods of $\infty$ are by definition the complements of the compact subsets of $\mathbb{N}$. The compact subsets of $\mathbb{N}$ are the finite subsets, so the neighbourhoods of $\infty$ are the sets with finite complement.

The map $n \to 1/n$ (where $1/\infty$ is interpreted as 0) takes this space to the set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$, and it is easy to check that it preserves the topology (because the points $1/n$ are all isolated, and the neighbourhoods of 0 are exactly the sets with finite complement). Observe that $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ is a compact and Hausdorff space, therefore, it is a one-point compactification of the subspace $\{1/n : n \in \mathbb{N}\}$.

**Question 5.** (*a*) State Tychonoff's Theorem. (*b*) Prove that the countable product (with the product topology), $[0,1]^w$, of the closed interval $[0,1]$, is sequentially compact.

(*a*) **Tychonoff's Theorem:** The product of any collection of compact topological spaces is compact with respect to the product topology.

(*b*) Let $X = [0,1]^w$ and $\tau$ is the product topology on $X$. Then $T = (X, \tau)$ is first-countable as the collection of all products $\prod_{n \in \mathbb{N}} U_n$ form a countable basis, where $U_n$ is an open interval with rational endpoints for finitely many values of $n$, and $U_n = \mathbb{R}$ for all other values of $n$. On the other hand, by Tychonoff's Theorem, $T$ is compact. Now, we will show that $T$ is sequentially compact, i.e., every infinite sequence in $X$ has a convergent subsequence.

Let $\langle x_n \rangle_{n \in \mathbb{N}}$ be an infinite sequence in $X$. We use a proof by contradiction. That is, suppose that $\langle x_n \rangle_{n \in \mathbb{N}}$ does not have a convergent subsequence. By Theorem 30.1, it follows that $\langle x_n \rangle_{n \in \mathbb{N}}$ does not have an accumulation point in $X$. Thus, for each $x \in X$, we can select an open set $U_x$ such that $x \in U_x$ and $U_x$ only contains $x_n$ for finitely many $n \in \mathbb{N}$. The set $\mathcal{U} = \{U_x : x \in X\}$ is clearly an open cover for $X$. By the definition of a compact space, there exists a finite subcover $\{U_{x_1}, U_{x_2}, \ldots, U_{x_m}\}$ of $\mathcal{U}$ for $X$. Then $U_{x_1} \cup U_{x_2} \cup \cdots \cup U_{x_m}$ contains all of $X$ since it is a cover. However, each open set in this union only contains $x_n$ for a finite number of $n$, and this is a finite union of sets. So, the union only contains $x_n$ for finitely many $n$. This implies that $x_n \in X$ for only finitely many $n \in \mathbb{N}$. This is a contradiction, since $\langle x_n \rangle$ is a sequence in $X$, and so $x_n \in X$ for all $n \in \mathbb{N}$.